Lesson 2. Review – Conditional Probability

1 Random variables

- A **random variable** is a variable that takes on its values by chance
    - One perspective: a random variable represents unknown future results
- Notation convention:
    - Uppercase letters (e.g. X, Y, Z) to denote random variables
    - Lowercase letters (e.g. x, y, z) to denote real numbers
- $\{X \leq x\}$ is the event that the random variable X is less than or equal to the real number x
- The probability this event occurs is written as $\Pr\{X \leq x\}$
- A random variable is **discrete** if it can take on only a finite or countably infinite number of values
    - e.g., X can take on the values $2, 4, 7, 9, 10$
- A random variable is **continuous** if it can take on a continuum of values
    - e.g., X can take on all nonnegative real values

2 Joint probabilities

- Let X and Y be discrete random variables:
    - X takes values $a_1, a_2, \ldots$
    - Y takes values $b_1, b_2, \ldots$
- X and Y could be **dependent**: for example,
    - X = service time of first customer in the shop
    - Y = delay (time in queue) of second customer in the shop
- The probability that $X = x$ and $Y = y$ is:

- We can obtain the **marginal probabilities** as follows:

Example 1. The Markov Company sells three types of replacement wheels and two types of bearings for in-line skates. Wheels and bearings must be ordered as a set, but customers can decide which combination of wheel type and bearing type they want.

Let V and W be random variables that represent the type of bearing and wheel, respectively, in replacement sets ordered in the future. Based on historical data, the company has determined the probability that each wheels-bearings pair will be ordered:

		W		
		1	2	3
V	1	2/10	1/10	1/10
	2	1/20	8/20	3/20

a. What is $\Pr\{V = 1 \text{ and } W = 2\}$?
b. What is $\Pr\{V = 1\}$?
c. What is $\Pr\{W = 2\}$?

3 Independence

- Let's consider events of the form $\{X \in \mathcal{A}\}$ and $\{Y \in \mathcal{B}\}$, for example:
  - $\mathcal{A} = (5, 27] \Rightarrow \{X \in \mathcal{A}\} =$
  - $\mathcal{B} = \{44, 73\} \Rightarrow \{Y \in \mathcal{B}\} =$
  - $\{X = a\}$ can be written as $\{X \in \mathcal{A}\}$ with
  - $\{Y \leq b\}$ can be written as $\{Y \in \mathcal{B}\}$ with
- Two random variables X and Y are **independent** if knowing the value of X does not change the probability of Y (and vice versa)
- Mathematically speaking, X and Y are independent if

Example 2. Are the random variables V and W in Example 1 independent? Why or why not?

4 Conditional probability

- **Conditional probability** addresses the question:

  How should we revise our probability statements about Y given that we have some knowledge of the value of X?

- The conditional probability that Y takes a value in $\mathcal{B}$ given that X takes a value in $\mathcal{A}$ is:

  - The revised probability is the probability of the joint event $\{Y \in \mathcal{B}, X \in \mathcal{A}\}$ normalized by the probability of the conditional event $\{X \in \mathcal{A}\}$

Example 3. In Example 1, what is the probability that a customer will order type 2 wheels, given that they order type 1 bearings? How about type 1 wheels? Type 3 wheels?

- If X and Y are independent, then:

- Let $\mathcal{A} \subseteq \mathcal{B}$. Then, if X and Y are perfectly dependent (i.e., $X = Y$), then:

5 Law of total probability

- We can write a joint probability as the product of a conditional probability and a marginal probability:

- Using this, we can also decompose a marginal probability into the products of conditional and marginal probabilities

- **The law of total probability.** Suppose X is a discrete random variable taking values $a_1, a_2, \ldots$. Then:

Example 4. In Example 1, the conditional probabilities of W given that $V = 2$ are:

b	1	2	3
$\Pr\{W = b \mid V = 2\}$	1/12	8/12	3/12

Use this with your answer to Example 3 to find $\Pr\{W = 2\}$.

6 Exercises

Problem 1. Professor I. M. Right often has his facts wrong. Let X be a random variable that represents the number of questions he is asked during one class, and let Y be the number of questions that he answers incorrectly during one class. The joint probabilities of X and Y are:

		Y = 0	Y = 1	Y = 2
	1	1/3	0	0
X	2	1/4	1/12	0
	3	3/16	1/8	1/48

a. What is the probability that Professor Right answers all questions correctly during one class?

b. What is the probability that Professor Right answers 1 question incorrectly during one class, given that he is asked two questions?

c. Explain why $\Pr\{X = 1 \text{ and } Y = 2\} = 0$.

Problem 2. The Simplex Company uses three machines to produce a large batch of similar manufactured items. 20% of the items were produced by machine 1, 30% by machine 2, and 50% by machine 3. In addition, 1% of the items produced by machine 1 are defective, 2% by machine 2 are defective, and 3% by machine 3 are defective. Suppose you select 1 item at random from the entire batch.

a. Define the random variable M as the machine used ($M \in \{1, 2, 3\}$) to produce this item. What is the probability that $M = m$, for $m = 1, 2, 3$?

b. Define another random variable D that is equal to 1 if this item is defective, and 0 otherwise. Find the probability that $D = 1$ given $M = m$, for $m = 1, 2, 3$.

c. Find the probability that $D = 1$; that is, the probability that the randomly selected item is defective.

Problem 3. Simplex Pizza sells pizza (of course) and muffins (that's weird). Let Z and M be random variables that represent the number of pizzas and muffins in one order, respectively. Based on historical data, the company has determined the joint probabilities for Z and M:

		M = 0	M = 1	M = 2
	0	0	0.09	0.06
Z	1	0.25	0.11	0.05
	2	0.10	0.08	0.07
	3	0.08	0.07	0.04

a. What is the probability that $M = m$, given that $Z = 2$, for $m = 0, 1, 2$?

b. What is the expected number of muffins in an order, given that it contains 2 pizzas?

c. It turns out that $\Pr\{M = 1\} = 0.35$ and $\Pr\{M = 1 \mid Z = 3\} \approx 0.368$. Based on this information, are M and Z independent? Why or why not?

Problem 4 (SMAS Exercise 3.6). In professional sports, players are often traded from team to team. Occasionally, trades are made without naming all of the players that will be traded, in which case the trade invovles "a player to be named later." In other words, the final result is uncertain.

Suppose there are 4 "players to be named later", who we will call 1, 2, 3, 4. Let X represent this player. The probabilities of X are:

a	1	2	3	4
$\Pr\{X = a\}$	0.1	0.3	0.5	0.1

a. What is the probability that player 4 will be traded, given that player 1 is not traded?
b. What is the probability that player 4 will be traded, given that players 1 and 2 are not traded?
c. What is the probability that player 2 will be traded, given that player 1 or 2 is traded?

Problem 5 (SMAS Exercise 3.9). A machine produces components that are either acceptable or defective. Let X_1 represent the first component and X_2 represent the second component, where a value of 0 corresponds to an acceptable component, and a value of 1 corresponds to a defective component. The joint probabilities of X_1 and X_2 are:

		X_2	
		0	1
X_1	0	0.75	0.05
	1	0.10	0.10

a. What is the probability that the second component will be defective if the first component is not defective?
b. What is the probability that the second component will be defective if the first component is defective?
c. Are X_1 and X_2 dependent?
d. If we make a profit of $100 for each acceptable component but lose $20 for each defective component produced, what is the expected profit for the second component given that the first component was defective?